Riemann's Rearrangement Theorem

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Since that ancient time when Zeno first sent Achilles chasing after the tortoise, infinite series have been a source of wonder and amusement because they can be manipulated to appear to contradict our understanding of numbers and nature. Zeno's paradoxes still intrigue and baffle us even though the fallacies in his arguments have long since been identified.

Mathematicians of the late seventeenth and eighteenth centuries were often puzzled by the results they would get while working with infinite series. By the nineteenth century it had become apparent that divergent series were often the cause of the difficulties. "Divergent series are the invention of the devil," Neils Hendrik Abel wrote in a letter to a friend in 1826. "By using them, one may draw any conclusion he pleases, and that is why these series have produced so many fallacies and so many paradoxes" (Kline 1972).

As an example of what can go wrong, suppose we let S represent the sum of the alternating harmonic series, that is

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

See figure 1. What's wrong here? (This series, by the way, is not divergent. Its sum is ln 2, which is easy to determine. Find the Taylor series of $\ln(1 + x)$ and let x equal 1.) It seems that although we merely rearranged the terms of an infinite series (equations 3 and 4), its sum has changed from $2S$ to S!

Write out a few terms:

(1) $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \cdots$

Multiply both sides by 2:

(2) $2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \cdots$

Collect terms with the same denominator, as the arrows indicate:

(3) $2S = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \frac{2}{7} - \frac{1}{4} + \frac{2}{9} - \frac{1}{5} + \frac{2}{11} - \frac{1}{6} + \cdots$

We arrive at this:

(4) $2S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$

We see that on the right side of this equation, we have the same series we started with. In other words, by combining equations 1 and 4, we obtain

(5) $2S = S.$

Divide by S. We have shown that

(4) $2 = 1.$

Fig. 1

In 1827, Peter Lejeune-Dirichlet discovered this surprising result while working on conditions that ensured the convergence of Fourier series. He was the first to notice that it is possible to rearrange the terms of certain series (now known as conditionally convergent series) so that the sum would change. Why is this result possible? Dirichlet was never able to give an answer. (In a paper published in 1837, he did prove that rearranging the terms of an absolutely convergent series does not alter its sum.) With the discovery that the sum of a series could be changed, Dirichlet had found the path to follow to prove the convergence of Fourier series. By 1829 he had succeeded in solving one of the preeminent problems of that time.

In 1852, Bernhard Riemann began work on a paper extending Dirichlet's results on the convergence of Fourier series. Riemann sought Dirichlet's advice and showed him a draft of this work. Dirichlet reminisced about his work on the problem and related his discovery that rearranging the terms of a conditionally convergent series could alter its sum. Riemann suspected that divergent series were somehow responsible. He soon found a remarkable explanation that accounted for this bizarre behavior, now known as Riemann's rearrangement theorem, which he incorporated in his paper on Fourier series. Although the paper was completed by the end of 1853, it was not published until after his death in 1866 under the title "On the Representation of a Function by a Trigonometric Series."

To get at Riemann's theorem we will use the definition of the sum of an infinite series and seven theorems that are part of a standard first course on infinite series. We will also need to distinguish between two types of convergent series.

The Sum of an Infinite Series

When confronting an infinite series for the first time, students are usually puzzled by what is meant by its sum. For instance, when considering the geometric series

$$(1) \qquad \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots ,$$

students often challenge the claim that the sum of this series is 1. "That series doesn't add up to 1," they often say. "Take any number of terms, say the first trillion, add them up, and you're not going to get 1. No matter how many terms you add on, the sum never reaches 1."

The observation is correct; the conclusion is wrong. What they don't notice is that they haven't added all the terms! Those weren't infinite series they were adding. Those were finite series. Before we can expect to obtain 1 as the sum, we need to add all the terms. The question is, How can we do so? How can we add an infinite number of terms? The answer is we don't directly add all the terms. Instead we look for a method that will allow us to see what number (if any) we would get if we were somehow able to perform the impossible task of adding up all the terms. In our search for the sum of the geometric series in (1), we look at the nth partial sum,

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i$$

as $n \to \infty$.

We see that

$$S_1 = \sum_{i=1}^{1} \left(\frac{1}{2}\right)^i = \frac{1}{2},$$

$$S_2 = \sum_{i=1}^{2} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = \frac{2^2 - 1}{2^2},$$

$$S_3 = \sum_{i=1}^{3} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = \frac{2^3 - 1}{2^3},$$

$$S_4 = \sum_{i=1}^{4} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16} = \frac{2^4 - 1}{2^4},$$

and so on. The nth partial sum is

$$S_n = \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n}.$$

As we add on more and more terms as n approaches infinity, the partial sums get closer and closer to 1. It seems that if we could somehow add on all the remaining terms, we would get 1 as the sum. So it appears that a plausible sum for this geometric series would be the limit of its nth partial sum as n approaches infinity,

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^n = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{1}{2}\right)^i = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

Therefore, it seems natural to define the sum of an infinite series as follows:

Definition. *If the limit of the nth partial sum S_n of the infinite series $\Sigma\, a_n$ exists and equals S, then we say $\Sigma\, a_n$ converges and its sum is S. If, as n approaches infinity, the limit of the nth partial sum S_n does not exist, then we say the series $\Sigma\, a_n$ diverges and has no sum.*

Convergence Tests

We were able to determine that the geometric series converges by examining its *n*th partial sum. However, as it turns out, a formula for the *n*th partial sum of most infinite series cannot be found. Augustin-Louis Cauchy, as well as Abel and Dirichlet, realized this difficulty and was among the first to devise a number of theorems or tests to determine the convergence of a series. Seven theorems on convergent and divergent series follow. Their proofs are relatively simple and rely heavily, as one would expect, on the definition of the sum of an infinite series. The proofs of these theorems can be found in practically any first-year calculus text.

Theorem 1. *The sum of two convergent series is a convergent series. If $\Sigma\, a_n = S$ and $\Sigma\, b_n = T$ then*

$$\sum (a_n + b_n) = S + T.$$

Theorem 2. *The sum of a convergent series and a divergent series is a divergent series.*

Theorem 3. $\sum_{1}^{\infty} a_n$ *and* $\sum_{i}^{\infty} a_n$ *both converge or both diverge. (In other words, the first finite number of terms do not determine the convergence of a series.)*

Theorem 4. If the series $\Sigma\, a_n$ converges, then $\lim_{n\to\infty} a^n = 0$.

Theorem 5. If $\Sigma\, |a_n|$ converges, then $\Sigma\, a_n$ converges.

Theorem 6. *The comparison test. If the series $\Sigma\, a_n$ and $\Sigma\, b_n$ have only positive terms with $a_n \le b_n$ for all $n \ge 1$, and*

(1) if $\Sigma\, b_n$ converges, then $\Sigma\, a_n$ converges;

(2) if $\Sigma\, a_n$ diverges, then $\Sigma\, b_n$ diverges.

Theorem 7. *Leibniz' alternating series test. The alternating series $\Sigma\, (-1)^{n-1} a_n$ converges if the sequence $\{a_n\}$ is monotone decreasing to 0.*

The Harmonic Series

Theorem 4 says that it is necessary for the terms of a series to approach 0 if the series is to converge. But is this a sufficient condition for a series to converge? The answer to this question is supplied by a rather famous counterexample, the harmonic series $\Sigma\,(1/n)$. The fact that the terms of the harmonic series going to 0 does not prevent the series from diverging can be shown by using the comparison test (Cauchy's integral test, which is another form of the comparison test, would provide an alternate method of proof). The terms of $\Sigma\,(1/n)$ can be grouped (not rearranged) as in figure 2.

Clearly each group sectioned off in the harmonic series is greater than $1/2$. So, in effect, we are summing a series in which every term is at least $1/2$; thus the nth partial sum S_n increases without bound, and the harmonic series must diverge. The divergence happens very slowly—approximately 2^{15} terms must be added before S_n exceeds 10, and approximately 2^{29} terms are needed before S_n exceeds 20.

$$
\begin{aligned}
\sum \frac{1}{n} &= 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{3} + \frac{1}{4}\right)}_{\substack{\text{2 terms} \\ \text{each } \geq\ 1/4}} + \underbrace{\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)}_{\substack{2^2 \text{ terms} \\ \text{each } \geq\ 1/8}} + \underbrace{\left(\frac{1}{9} + \cdots + \frac{1}{16}\right)}_{\substack{2^3 \text{ terms} \\ \text{each } \geq\ 1/16}} + \cdots \\
&> 1 + \frac{1}{2} + \underbrace{\left(\frac{1}{4} + \frac{1}{4}\right)}_{} + \underbrace{\left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)}_{} + \underbrace{\left(\frac{1}{16} + \cdots + \frac{1}{16}\right)}_{} + \cdots \\
&= 1 + \frac{1}{2} + \qquad \frac{1}{2} \qquad + \qquad \frac{1}{2} \qquad + \qquad \frac{1}{2} \qquad + \cdots
\end{aligned}
$$

Fig. 2

The alternating harmonic series

$$\sum \frac{(-1)^{n-1}}{n}$$

is a different story. The absolute value of the terms of this series are monotonic decreasing to 0. By an argument made famous by Leibniz (the alternating-series test), we can conclude that the alternating harmonic series converges.

So we see that although the alternating harmonic series converges, the series obtained by replacing each term by its absolute value diverges. This result shows that the convergence of $\Sigma\, a_n$ does not imply the convergence of $\Sigma\, |a_n|$.

Two Types of Infinite Series

Considering the harmonic series, the alternating harmonic series, and theorem 5, we are led naturally to define two types of infinite series. A series $\Sigma\, a_n$ is called *absolutely convergent* if $\Sigma\, |a_n|$ converges. A series is called *conditionally convergent* if $\Sigma\, a_n$ converges but $\Sigma\, |a_n|$ diverges.

Is the alternating harmonic series an absolutely or conditionally convergent series? If we take the absolute value of all the terms, we get the harmonic series, which, as we have seen, diverges. So the alternating harmonic series is a conditionally convergent series.

However, the series

$$\sum \frac{(-1)^{n-1}}{n^2}$$

is an absolutely convergent series, since when we take the absolute value of all its terms we get $\Sigma\,(1/n^2)$, and this series is known to converge. We can show that $\Sigma\,(1/n^2)$ converges by the comparison test. (Once again, Cauchy's integral test could be used instead.)

$\Sigma\,(1/n^2)$ is term by term less than

$$\text{(2)} \qquad \sum \frac{1}{n(n-1)}$$

for all $n > 1$. If we can show that (2) converges, then since it dominates $\Sigma\,(1/n^2)$, we can conclude that $\Sigma\,(1/n^2)$ converges:

$$\begin{aligned} \sum_{2}^{\infty} \frac{1}{n(n-1)} &= \sum_{2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= \lim_{n\to\infty} \sum_{2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) \\ &= \lim_{n\to\infty} \left(1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \cdots \right. \\ &\qquad \left. - \frac{1}{n-1} + \frac{1}{n-1} - \frac{1}{n} \right) \end{aligned}$$

(from writing out a few of the first and last terms)

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n} \right)$$

(We see that the finite series telescopes.)

$$= 1$$

Since (2) converges, it follows that $\Sigma\,(1/n^2)$ converges. To summarize,

$$\sum \frac{(-1)^{n-1}}{n}$$

is conditionally convergent because

$$\sum \left|\frac{(-1)^{n-1}}{n}\right| = \sum \frac{1}{n}$$

diverges.

$$\sum \frac{(-1)^{n-1}}{n^2}$$

is absolutely convergent because

$$\sum \left|\frac{(-1)^{n-1}}{n^2}\right| = \sum \frac{1}{n^2}$$

converges.

This conclusion brings us to Riemann's rearrangement theorem, which will be presented in two parts.

Riemann's Rearrangement Theorem—Part 1

In a conditionally convergent series, the sum of the positive terms is a divergent series and the sum of the negative terms is a divergent series.

Proof. First we notice that a conditionally convergent series must have positive and negative terms. If all its terms were positive or all were negative, it would be an absolutely convergent series. For example, if $\Sigma\, b_n = -1$ and all the terms b_n were negative, then $\Sigma\, |b_n|$ would converge to 1 and $\Sigma\, b_n$ would be an absolutely convergent series.

In fact, conditionally convergent series must have an infinite number of positive and negative terms. If $\Sigma\, b_n$ has only a finite number of negative terms, then the remaining series of positive terms must converge, since by theorem 3 the first finite number of terms do not count when we determine the convergence or divergence of an infinite series. This result would mean that $\Sigma\, b_n$ is an absolutely convergent series.

So we take a conditionally convergent series $\Sigma\, a_n$ and separate it into two infinite series, one of all the positive terms and the other of all the negative terms, and represent these series by Σa_n^+ and Σa_n^-, respectively. So that we can successfully recover the original series without rearranging terms by writing $\Sigma\, a_n = \Sigma\, a_n^+ + \Sigma\, a_n^-$, we define the terms a_n^+ and a_n^- as follows:

$$a_n^+ = \begin{cases} a_n & a_n > 0 \\ 0 & a_n < 0 \end{cases}$$

$$a_n^- = \begin{cases} 0 & a_n > 0 \\ a_n & a_n < 0 \end{cases}$$

For instance, for the series

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \ - \cdots,$$

$$\sum a_n^+ = 1 + 0 + 0 + \frac{1}{4} + 0 + 0 + \frac{1}{7} + \cdots$$

and

$$\sum a_n^- = 0 - \frac{1}{2} - \frac{1}{3} + 0 - \frac{1}{5} - \frac{1}{6} + 0 - \cdots.$$

So

$$\sum a_n = \sum a_n^+ + \sum a_n^- .$$

For the convergence of $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$, four possibilities exist:

Case 1: $\Sigma\, a_n^+$ converges and $\Sigma\, a_n^-$ converges.

Case 2: $\Sigma\, a_n^+$ converges and $\Sigma\, a_n^-$ diverges.

Case 3: $\Sigma\, a_n^+$ diverges and $\Sigma\, a_n^-$ converges.

Case 4: $\Sigma\, a_n^+$ diverges and $\Sigma\, a_n^-$ diverges.

Using the definitions of absolutely and conditionally convergent series, Riemann showed that cases 1, 2, and 3 are impossible and that, hence, case 4 follows.

Here is how he did it. We can't have case 1, for suppose

$$\sum a_n^+ = S$$

and

$$\sum a_n^- = -T$$

with $T > 0$. Then

$$\sum |a_n^-| = T.$$

In this case, since

$$\sum |a_n| = \sum a_n^+ + \sum |a_n^-|,$$

$\Sigma\, |a_n|$ is the sum of two convergent series. Therefore by theorem 1, $\Sigma\, |a_n|$ must converge to $S + T$. This result means that $\Sigma\, a_n$ would be an absolutely convergent series, not conditionally convergent as required.

We can't have case 2, because if we add both the convergent series $\Sigma\, a_n^+$ and the divergent series $\Sigma\, a_n^-$, the resulting series $\Sigma\, a_n$ will diverge, since by theorem 2 the sum of a convergent series and a divergent series is a divergent series. But we know that $\Sigma\, a_n$ is conditionally convergent and hence must converge.

For example, if

$$\sum a_n^+ = 1$$

and Σa_n^- diverges to $-\infty$, then it would follow that $\Sigma\, a_n$ diverges to $-\infty$, but $\Sigma\, a_n$ must converge because it is given to be a conditionally convergent series.

Case 3 is essentially the same as case 2. Therefore, we must have case 4, $\Sigma\, a_n^+$ and $\Sigma\, a_n^-$ are both divergent series for a conditionally convergent series. The divergence of the two series is the key idea in proving the second part of Riemann's rearrangement theorem. It offers an insight as to why the sum of a conditionally convergent series can be changed by rearranging terms. In fact, as we will now see, the terms can be rearranged to add up to any number we wish!

Riemann's Rearrangement Theorem—Part 2

Let $\Sigma\, a_n$ be a conditionally convergent series, and let S be a given real number. Then a rearrangement of the terms of $\Sigma\, a_n$ exists that converges to S.

Proof. We wish to show we can rearrange the terms of $\Sigma\, a_n$ to form a series whose sum is S.

Add together, in order, just enough of the first positive terms of $\Sigma\, a_n$ so that their sum exceeds S. Say we need n_1 terms to do so, then the partial sum $S_{n_1} > S$. We can always do so no matter how large S is, since the series of positive terms diverges to infinity.

To this sum S_{n_1} add, in order, just enough of the first negative terms of $\Sigma\, a_n$ to make the resulting sum less than S. Say we need n_2 negative terms, then the partial sum

$$S_{n_1+n_2} < S.$$

We can always do so no matter how far the positive terms took us to the right of S, since the series of negative terms diverges to negative infinity.

Now add just enough of the next positive terms, say n_3 positive terms, to get the sum to exceed S again. We now have the partial sum

$$S_{n_1+n_2+n_3} > S.$$

Now again add just enough of the next negative terms, say n_4 negative terms, so that the sum is less than S again. We now have the partial sum

$$S_{n_1+n_2+n_3+n_4} < S.$$

We continue to repeat this process, adding each time just enough new positive terms to make the sum exceed S and then just enough new negative terms to make the sum less than S.

Now we notice that all these partial sums differ from S by, at most, one positive or one negative term. These partial sums must be closing in on S—since the original series $\Sigma\, a_n$ converges, its terms a_n go to 0 as n goes to infinity. We can get the partial sums as close to S as we wish (within any epsilon), if we go out far enough in the series (to where all the remaining terms have their absolute values less than epsilon). So the partial sums converge to S, which proves that the series of rearranged terms converges to S.

It should be pointed out that a rearrangement need not be constructed in the manner just described. Rearrangements are not unique. For instance, suppose we wish the sum of a conditionally convergent series to be 10. We could add the first positive terms to a million or a billion and be completely confident that we could add on negative terms and get back to 10, since the series of negative terms diverges. If we somehow lost track of what we were doing and continued to add on negative terms to $-1{,}000{,}000{,}000{,}000$, we need not worry because at any time, we can begin to add on positive terms and get the partial sums back to within any range of 10 that we wish. We can swing back and forth to numbers larger than we have ever imagined, just as long as we eventually decide to bring the partial sums back to oscillate around 10.

We can also rearrange the terms of any conditionally convergent series so that it will diverge. One such rearrangement is to pick positive terms to add to a million, then add on one negative term, then add on positive terms to reach a trillion, then add on another negative term, then add positive terms till we are beyond a googolplex, then add on a negative term . . . .

That's Riemann's rearrangement theorem. It's really a grand counterexample to to the seemingly plausible idea that we can rearrange the terms of any infinite series and be sure that we will not alter its sum.

The proof of Riemann's theorem offers a model for getting partial sums within an epsilon of the sum of an infinite series and uses the idea that the terms of a convergent series must go to 0, along with other basic theorems of a first course. The theorem gives students a deeper understanding of infinite series and can easily be proved to an Advanced Placement class without sacrificing the teaching of other material.

The final point to be made here is that Riemann's rearrangement theorem is a counterexample that surprises and even startles us. It challenges our conceptions in an interesting way. Seeing what can go wrong is often an indispensable way to gain insight and intuition into the foundations of calculus.

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